Non-abelian discrete flavor symmetries from T^2/Z_N orbifolds

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ABSTRACT: In [1] it was shown how the flavor symmetry A_4 (or S_4) can arise if the three fermion generations are taken to live on the fixed points of a specific 2-dimensional orbifold. The flavor symmetry is a remnant of the 6-dimensional Poincaré symmetry, after it is broken down to the 4-dimensional Poincaré symmetry through compactification via orbifolding. This raises the question if there are further non-abelian discrete symmetries that can arise in a similar setup. To this end, we generalize the discussion by considering all possible 2-dimensional orbifolds and the flavor symmetries that arise from them. The symmetries we obtain from these orbifolds are, in addition to S_4 and A_4, the groups D_3, D_4 and $D_6 \simeq D_3 \times Z_2$ which are all popular groups for flavored model building.

Contents

1 Introduction

The flavor problem of the Standard Model of Particle Physics has two aspects. First, the question arises what flavor is. Next, one can ask why the parameters of the flavor sector, the fermion masses and the mixing matrices, take the values they do. A popular and successful approach is to impose a non-abelian discrete flavor symmetry to explain certain observed regularities. The nature of flavor is, in the context of flavor symmetries, therefore usually reduced to the question as to the origin of that symmetry.

Two main types of symmetries are needed to construct the Lagrangian of the Standard Model: space-time and gauge symmetries. In general adding an additional gauge group to the Standard Model is a much simpler task than extending the space-time symmetry. However, breaking a continuous flavor gauge group down to a non-abelian discrete subgroup is a highly non-trivial phenomenological task. In particular, for such a breaking, large representations of the continuous symmetry are needed, which can not couple directly to the small representations in which the three generations of fermions would reside.[1] It is thus worthwhile to consider discrete flavor symmetries arising as extensions of the space-time symmetry.

An extension of the space-time symmetry can only be achieved by an extension of space-time itself. We thus need to work in an extra-dimensional framework. Such an extension of space-time will enlarge the Poincaré symmetry. If the n extra dimensions are compactified in an orbifold, the space-time symmetry will not be the full $4 + n$-dimensional

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[1]For further details we refer to [2].

Poincaré symmetry. However, depending on the exact compactification, there may be residual discrete symmetries, which can then play the role of flavor symmetry.

This idea was first explored in [1], where two extra dimensions were assumed. This can be considered the minimal number in this setup, as one extra dimension does not lead to non-abelian discrete symmetries. For a specific 2-dimensional orbifold it was shown there, that the residual Poincaré symmetry is the group S_4, the group of permutations of four distinct objects (if discrete symmetries, such as parity, are not taken into account, i.e. if we only consider proper Lorentz transformations, the residual symmetry is A_4). A_4 [3] and S_4 [4] are both popular and phenomenologically successful as flavor symmetries, especially for predicting tri-bimaximal neutrino mixing. In this paper we generalize the discussion of [1] by considering all possible 2-dimensional orbifolds and calculating the resulting symmetry. As it turns out, the resulting flavor symmetries are, in addition to A_4 and S_4, the three dihedral groups D_4, $D_3 \cong S_3$ and $D_6 \cong D_3 \times Z_2$, all of which have been widely used as flavor symmetries [5–7].

Another way of obtaining discrete flavor symmetries from orbifolds is inspired by string theory and uses string selection rules [8]. We will not be using this approach and will only be employing regular field theory on an orbifold. However, as discussed in [8], the two approaches do not contradict each other: If we have an orbifold possessing an inherent discrete symmetry, such as the ones we discuss in this paper, and then also impose the string selection rules, we will end up with an enlarged flavor symmetry.

This paper is organized as follows. In section 2 we discuss the possible 2-dimensional orbifolds and review how the discrete symmetries can be extracted from them. We also explain, why a 1-dimensional orbifold is not sufficient to obtain a non-abelian flavor symmetry. In section 3 we discuss orbifold by orbifold which symmetry group arises from it. In section 4 we discuss the relation between flavor group representations and brane fields constrained to the fixed points in a certain twisted sector. Finally we conclude in section 5.

2   Orbifolding

We work in a 6-dimensional framework, where the two extra dimensions are compactified on an orbifold T^2/Z_N [9]. The co-ordinates in the two extra dimensions are denoted by (x_5, x_6).

The 2-dimensional torus T^2 is obtained by identifying the opposite sides of a parallelogram:

$$(x_5, x_6) \rightarrow (x_5, x_6) + \vec{e}_1$$
$$(x_5, x_6) \rightarrow (x_5, x_6) + \vec{e}_2 \ , \tag{2.1}$$

where $\vec{e}_1 = (1,0)$, $\vec{e}_2 = C(\cos(\alpha), \sin(\alpha))$ are the basis vectors of the torus. We can always choose $\vec{e}_1$ to point along the x_5 axis and to be normalized, leaving two free parameters defining $\vec{e}_2$, C and α, the length and the angle with respect to the x_5 axis. In this torus, the origin $(0,0)$ is identified with all points of the form

$$a\vec{e}_1 + b\vec{e}_2 \ , \tag{2.2}$$

where a, b are integers.

Aside from the torus basis, the orbifold is further defined by the abelian group Z_N which is modded out of the torus. This means that we further identify points related by a rotation around the origin through integer multiples of an angle ϕ, with $N\phi = 2\pi$. The choice of Z_N is strictly constrained, as we discuss in the following [10]. The group Z_N is generated by one element, which corresponds to a rotation by the angle ϕ. Its matrix representation in the Cartesian x_5-x_6 basis is thus

$$\omega = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} . \tag{2.3}$$

Since the origin does not change under the rotation, all the points which are identified with the origin in the torus should be rotated to points which are also identified with the origin, i.e.

$$\omega(a\vec{e}_1 + b\vec{e}_2) = a'\vec{e}_1 + b'\vec{e}_2 , \tag{2.4}$$

where a, a', b and b' are all integers.

Instead of using Cartesian coordinates, we can use the torus basis $\vec{e}_1, \vec{e}_2$. The matrix representation of the generating element in this basis reads

$$\hat{\omega} = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} , \tag{2.5}$$

where $\hat{\omega} = U\omega U^{-1}$ and U is the similarity transformation relating the Cartesian and Torus bases to each other. In this basis we have

$$\begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} . \tag{2.6}$$

Due to the fact that a, b, a', b' are integers, the n_i must also be integers. And since the trace is a basis-independent quantity, we have

$$2\cos(\phi) = Tr\omega = Tr\hat{\omega} = n_1 + n_4 , \tag{2.7}$$

which implies that $2\cos(\phi)$ is an integer and thus $\cos(\phi) = -1, -1/2, 0, 1/2, 1$ corresponding to $\phi = \pi, 2\pi/3, \pi/2, \pi/3, 2\pi$. This directly leads to a constraint for the Z_N, and we are only allowed to choose $N = 1, 2, 3, 4, 6$. This then also leads to a constraint concerning our choice of torus basis vectors, since the rotational symmetry Z_N needs to be consistent with the symmetry of the torus. When modding out Z_2, this is no constraint, as any basis is consistent with reflections. For Z_3 and Z_6 we can only take the relative angle between the basis vectors to be 60, 120 or 150 degrees. All three possibilities give the same orbifold. In this paper, we choose the 60° lattice with basis vectors $(\vec{e}_1 = (1, 0), \vec{e}_2 = (1/2, \sqrt{3}/2))$.[2] Finally, when modding out Z_4 the only possibility is a 90° lattice, with both basis vectors normalized to a length of 1.

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[2] The other two equivalent possibilities are the SU(3) lattice with $(\vec{e}_1 = (1, 0), \vec{e}_2 = (-1/2, \sqrt{3}/2))$ and the G_2 lattice with $(\vec{e}_1 = (1, 0), \vec{e}_2 = (-3/2, \sqrt{3}/2))$.

We thus only have to discuss four different cases: T^2/Z_2, T^2/Z_3, T^2/Z_4 and T^2/Z_6. For the last three, the orbifold is uniquely defined, while for the first case we need to additionally discuss the effect of choosing a specific basis.

From these four orbifolds, we can then extract the residual Poincaré symmetry, which will in all cases be a non-abelian discrete symmetry. This is done in the following way: After choosing the orbifold, we determine the fixed points. A fixed point is a point for which a rotation by an integer multiple of ϕ is equivalent to a lattice translation. These points are potential candidates for the localization of 3-branes[3] and thus the Standard Model fermions can be taken to be brane fields, which are non-vanishing only at the fixed points. The fixed points are divided into several twisted sectors, where the mth twisted sector contains those fixed points for which a rotation by $m\phi$ corresponds to a lattice translation. A given fixed point can lie in several twisted sectors.

We assume all fixed points to be physically equivalent. This then means that the remnant translation and rotation symmetries are those which result only in a permutation of the fixed points, i.e. only map fixed points to other fixed points. These remnant symmetry operations are the elements of the residual Poincaré symmetry, and all that remains to be done is to find the underlying group structure.

One can then immediately see, why we do not need to consider the 1-dimensional orbifold S^1/Z_N: It has only two fixed points, and thus any symmetry group which permutes them will be a subgroup of the permutation group for two distinct objects, $S_2 \simeq Z_2$, which is abelian. Since we want to obtain a non-abelian discrete symmetry, we need to consider at least a 2-dimensional orbifold.

3 Symmetries from orbifolding

In our discussion we parametrize the two extra dimensions by a complex number $z \equiv x_5 + ix_6$. Analogously to equation (2.1), the Torus T^2 is obtained by identifying the points in the complex plane related by

$$z \to z + 1 , \tag{3.1}$$

$$z \to z + \gamma , \tag{3.2}$$

where the complex numbers $(1, \gamma)$ correspond to the basis vectors $(\vec{e}_1, \vec{e}_2)$.

3.1 T^2/Z_2

If we mod out a Z_2 reflection symmetry, γ can be arbitrary in general. However, in order to obtain a non-abelian symmetry, we have only two possibilities: The first one is $\gamma = e^{i\pi/3}$, which gives us an S_4 flavor symmetry, or an A_4 symmetry if only proper Lorentz transformations and translations (i.e. no discrete parities) are considered. The other possible basis is $\gamma = e^{i\pi/2} = i$. Since the case of S_4 and A_4 has already been discussed in [1], we will only discuss the case $\gamma = e^{i\pi/2} = i$ here. This orbifold is shown in

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[3]A 3-brane has three spatial dimensions.

figure 1. The Z_2 parity is defined by

$$z \rightarrow -z \ . \tag{3.3}$$

The fixed points are then given by $(z_1, z_2, z_3, z_4) = (1/2, (1+i)/2, i/2, 0)$. The fixed points are permuted by the two translation operations

$$S_1 : z \rightarrow z + 1/2 \ , \tag{3.4}$$

$$S_2 : z \rightarrow z + i/2 \ . \tag{3.5}$$

Moreover, the fixed points are also permuted by the rotation

$$T_R : z \rightarrow \omega z \ , \tag{3.6}$$

where $\omega = e^{i\pi/2} = i$. One can also write these operations explicitly in terms of the interchange of the fixed points,

$$S_1[(14)(23)] : (z_1, z_2, z_3, z_4) \rightarrow (z_4, z_3, z_2, z_1) \ , \tag{3.7}$$

$$S_2[(12)(34)] : (z_1, z_2, z_3, z_4) \rightarrow (z_2, z_1, z_4, z_3) \ , \tag{3.8}$$

$$T_R[(13)(2)(4)] : (z_1, z_2, z_3, z_4) \rightarrow (z_3, z_2, z_1, z_4) \ . \tag{3.9}$$

From these elements we can define two generators,

$$A = [(13)(2)(4)][(14)(23)] = [(1432)] \ , \tag{3.10}$$

$$B = [(12)(34)] \ , \tag{3.11}$$

satisfying the generator relations,

$$A^4 = 1 \ ,$$
$$B^2 = 1 \ ,$$
$$ABA = B \ . \tag{3.12}$$

This describes the dihedral group D_4, the symmetry group of the square. The group theory of D_4, and of the dihedral groups in general, is discussed for example in [5]. Note that this group is not enlarged if we include parity transformations.

3.2 T^2/Z_3

When modding out Z_3 we consider, without loss of generality, only the torus with a 60° lattice, as already mentioned in section 2. This corresponds to the choice $\gamma = e^{i\pi/3}$. This orbifold is shown in figure 1. The operation of the generator of the Z_3 symmetry is given by

$$z \rightarrow e^{i2\pi/3}z \ . \tag{3.13}$$

The corresponding fixed points are $(z_1, z_2, z_3) = (0, i/\sqrt{3}, 1/2 + i/2\sqrt{3})$. The translation operations permuting these fixed points are

$$S_1 : z \rightarrow z + (1/2 + i/2\sqrt{3}) \ , \tag{3.14}$$

$$S_2 : z \rightarrow z + i/\sqrt{3} \ . \tag{3.15}$$
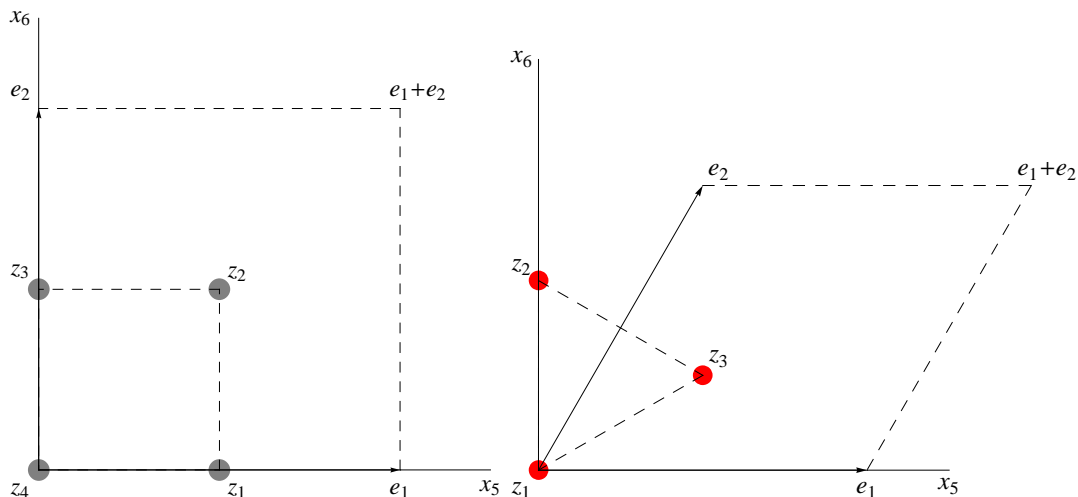
Figure 1. The orbifolds T^2/Z_2 (left) and T^2/Z_3 (right) with basis vectors $\vec{e_1}$, $\vec{e_2}$ and fixed points z_i. The square (left) and the triangle (right) formed by the fixed points, corresponding to the discrete symmetries D_4 and D_3 respectively, are shown with dashed lines.

Moreover, the fixed points are also permuted by the rotation with respect to the origin

$$T_R : z \rightarrow \omega z , \tag{3.16}$$

where $\omega = e^{i\pi/3} = i$. Again, one can also write the symmetry operations in terms of a permutation of the fixed points,

$$S_1[(321)] : (z_1, z_2, z_3) \rightarrow (z_2, z_3, z_1) , \tag{3.17}$$

$$S_2[(123)] : (z_1, z_2, z_3) \rightarrow (z_3, z_1, z_2) , \tag{3.18}$$

$$T_R[(23)] : (z_1, z_2, z_3) \rightarrow (z_1, z_3, z_2) . \tag{3.19}$$

A possible parity transformation would be equivalent to the rotation T_R and thus does not need to be considered separately. We can formulate two generators
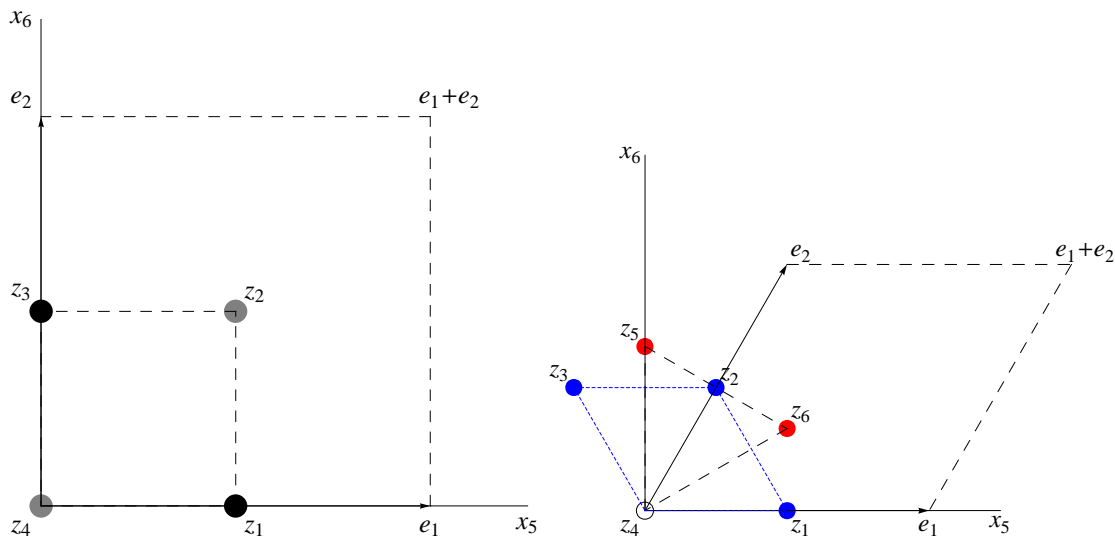
$$A = [(321)] , \tag{3.20}$$

$$B = [(321)][(23)] = [(13)] , \tag{3.21}$$

satisfying the generator relations,

$$A^3 = 1 ,$$
$$B^2 = 1 ,$$
$$ABA = B . \tag{3.22}$$

This describes the dihedral group D_3, the symmetry group of the triangle, which is isomorphic to S_3 the permutation group of three distinct objects. As it is also a dihedral symmetry, its group theory is discussed in more detail in [5].

Figure 2. The orbifolds T^2/Z_4 (left) and T^2/Z_6 (right) with basis vectors $\vec{e_1}$, $\vec{e_2}$ and fixed points z_i. On the left, the fixed points which are both in the first and the second twisted sector are designated by gray points, those fixed points which are only in the second twisted sector are designated by black points. On the right, the fixed point which is in all twisted sectors is represented by a circle, those fixed points which are only in the second twisted sector are designated by red (lighter gray) points, while those fixed points which are only in the third twisted sector are given by blue (darker gray) points.

3.3 T^2/Z_4

When modding out the abelian group Z_4, we have only one consistent choice of basis, $\gamma = e^{i\pi/2} = i$. The torus is the same one we used for T^2/Z_2 to obtain the D_4 symmetry, as one can also see in figure 2. In fact, the fixed points will also be the same and we thus obtain the same flavor symmetry. This is due to the fact that we obtain all fixed points of the orbifold T^2/Z_4 in the second twisted sector, where we only consider the squared generator of Z_4. This corresponds to a Z_2 subgroup of Z_4 and is thus fully equivalent to our discussion for T^2/Z_2 with a 90° lattice. The first twisted sector only contains the fixed points z_2 and z_4; as both of them also appear in the second twisted sector no new fixed points and thus no new residual translational or rotational symmetry operations arise due to the larger abelian group, Z_4. The unique symmetry we thus obtain is D_4.

3.4 T^2/Z_6

As for T^2/Z_3 we use the 60° lattice, i.e. $\gamma = e^{i\pi/3}$. The orbifold is shown in figure 2. The operation of the Z_6 symmetry for the first twisted sector is defined by

$$z \rightarrow e^{i2\pi/6} z \ . \tag{3.23}$$

For the first twisted sector, we have only one fixed point which is $z_4 = 0$. For the second twisted sector, the operation of the Z_6 symmetry reads

$$z \rightarrow e^{i2\pi/3} z \ . \tag{3.24}$$

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The fixed points of the second twisted sector are $(z_4, z_5, z_6) = (0, i/\sqrt{3}, 1/2(1+i/\sqrt{3}))$ which are the same as in the case of T^2/Z_3.

For the third twisted sector, finally, the operation of Z_6 symmetry is written as

$$z \to -z . \tag{3.25}$$

The fixed points in this sector thus are $(z_1, z_2, z_3, z_4) = (1/2, 1/4+i\sqrt{3}/4, -1/4+i\sqrt{3}/4, 0)$. Combining all fixed points $(z_1, z_2, z_3, z_4, z_5, z_6)$, we find that the fixed points are only permuted by residual rotation operations, i.e. translation symmetry is fully broken. These rotations are

$$T_{R1} : z \to e^{i\pi/3}z , \tag{3.26}$$

$$T_{R2} : z \to e^{i2\pi/3}z . \tag{3.27}$$

Moreover, if we assume the full Poincaré symmetry, we also have two parity operations acting on the fixed points

$$P_1 : z \to z^*, \tag{3.28}$$

$$P_2 : z \to -z^*, \tag{3.29}$$

where z^* denotes the complex conjugation of z.

We can write all of these symmetry operations in terms of a permutation of the fixed points as

$$T_{R1}[(123)(56)] : (z_1, z_2, z_3, z_4, z_5, z_6) \to (z_3, z_1, z_2, z_4, z_6, z_5) , \tag{3.30}$$

$$T_{R2}[(132)] : (z_1, z_2, z_3, z_4, z_5, z_6) \to (z_2, z_3, z_1, z_4, z_5, z_6) , \tag{3.31}$$

$$P_1[(23)(56)] : (z_1, z_2, z_3, z_4, z_5, z_6) \to (z_1, z_3, z_2, z_4, z_6, z_5) , \tag{3.32}$$

$$P_2[(23)] : (z_1, z_2, z_3, z_4, z_5, z_6) \to (z_1, z_3, z_2, z_4, z_5, z_6) . \tag{3.33}$$

From these operators, we can form the generators

$$A = [(123)(56)] , \tag{3.34}$$

$$B = [(23)] , \tag{3.35}$$

satisfying the generator relations,

$$A^6 = 1 ,$$
$$B^2 = 1 ,$$
$$ABA = B . \tag{3.36}$$

This defines the group $D_6 \cong D_3 \times Z_2 \cong S_3 \times Z_2$. If we do not include the parity operations, we effectively lose the generator B. The flavor symmetry then has only one generator and is the abelian group Z_6.

4 Group representations

To construct a full model, one now needs to assign the fermion generations to representations of these flavor groups. The orbifold fixed points are interpreted as 3-branes, on which the fermion fields are localized. The flavor symmetry operations which permute the fixed points then act non-trivially on the fermion fields. Irreducible representations correspond to relations among the field values at different fixed points; these relations are invariant under symmetry operations. In general this means that one or more fermion generations transforming under an irreducible representation of the flavor group will be "smeared out" over all available fixed points. All representations can be reproduced in this way, the origin of the flavor group from orbifolding thus does not offer any restrictions on the choice of representations. Also, all representations will correspond in general to the field(s) being non-vanishing at all fixed points. Thus, although the flavor symmetry as a whole has a straightforward interpretation in the geometry of the orbifold, the different representations do not.

This is at least a bit different for the last orbifold we have discussed, T^2/Z_6. The resulting flavor symmetry was D_6, which is isomorphic to $D_3 \times Z_2$. We observe that all symmetry operations leave the origin, the fixed point z_4, invariant. Thus a field which is localized at the origin will transform trivially under the flavor symmetry. In addition the subgroup D_3 generated by A^2 and B leaves the fixed points z_5 and z_6, i.e. the fixed points of the second twisted sector, invariant. Fields localized only on these two fixed points thus transform non-trivially only under the Z_2 factor of the flavor group. Similarly, the fixed points of the third twisted sector, z_1, z_2 and z_3 are not permuted by the group element A^3, which generates Z_2. Fields localized in this sector will thus only transform non-trivially under the D_3 factor of the flavor group. Fields transforming non-trivially both under D_3 and Z_2 will necessarily be non-vanishing in both the second and the third twisted sector. For more details on the representation theory of D_6 and the transformation properties of representations under subgroups, see [5].

The orbifold T^2/Z_6 thus offers the aesthetical appeal that different representations correspond to different localizations in the orbifold and therefore have a more intuitive interpretation in terms of the orbifold geometry. However also here all representations can be reproduced, and the orbifold origin of the flavor symmetry does not offer further input as to which representations to use for model building.

5 Conclusion

We have discussed all possible non-abelian discrete symmetries arising from 2-dimensional orbifolds. In this context the flavor symmetries arise as a remnant symmetry of the full 6-dimensional space-time symmetry. This remnant symmetry can then be interpreted as the permutation symmetry of the orbifold fixed points. These fixed points in turn are taken to be 3-branes, on which the three generations of Standard Model fermions reside. The flavor symmetry then has a straightforward interpretation in terms of the geometry of the orbifold. As in crystallography, the number of possible lattice structures and symmetry groups is

strictly limited for orbifolds. The resulting flavor symmetries are all crystallographic point groups, as was to be expected. The possible flavor groups we obtain are S_4, A_4, S_3, D_4, and $D_6 \simeq D_3 \times Z_2$, where the first two were already discussed in [1]. All of these groups have been widely used as phenomenologically successful flavor symmetries.

The question is of course what implications these findings have for flavored model building. We found that, in all cases, the representation content of such models is not limited by assuming an orbifold origin for the flavor symmetry. In the case of D_6 at least the representations can be interpreted in terms of localization of the fields in specific sectors, but without a full theory of why certain fields are localized where, this does not offer direct model building input either. One can thus take two major hints from this general approach: First of all, it favors the well-known, small crystallographic groups as flavor symmetries, due to the crystalline structure of the two extra dimensions. This is however due to the fact that we have only considered two extra dimensions. Larger, more complicated, flavor symmetries may arise when considering more extra dimensions. This leads us to the second point: Further input for model building requires an extended analysis of the extra-dimensional setup. This has been done in the context of string theory [8], where interestingly enough the flavor group D_4 also appears naturally. However, the results of this paper can also be combined with regular extra-dimensional field theory. For example, the flavor groups arising from the orbifolds need to be further broken, a process which may actually be intimately connected with the extra dimensions themselves [11].

Acknowledgments

References

[1] G. Altarelli, F. Feruglio and Y. Lin, *Tri-bimaximal neutrino mixing from orbifolding*, *Nucl. Phys.* **B 775** (2007) 31 [hep-ph/0610165] [SPIRES].

[2] A. Adulpravitchai, A. Blum and M. Lindner, *Non-abelian discrete groups from the breaking of continuous flavor symmetries*, arXiv:0907.2332 [SPIRES].

[3] E. Ma and G. Rajasekaran, *Softly broken A_4 symmetry for nearly degenerate neutrino masses*, *Phys. Rev.* **D 64** (2001) 113012 [hep-ph/0106291] [SPIRES];
K.S. Babu, E. Ma and J.W.F. Valle, *Underlying A_4 symmetry for the neutrino mass matrix and the quark mixing matrix*, *Phys. Lett.* **B 552** (2003) 207 [hep-ph/0206292] [SPIRES];
M. Hirsch, J.C. Romao, S. Skadhauge, J.W.F. Valle and A. Villanova del Moral, *Degenerate neutrinos from a supersymmetric A_4 model*, hep-ph/0312244 [SPIRES]; *Phenomenological tests of supersymmetric A_4 family symmetry model of neutrino mass*, *Phys. Rev.* **D 69** (2004) 093006 [hep-ph/0312265] [SPIRES];
E. Ma, *Quark mass matrices in the A_4 model*, *Mod. Phys. Lett.* **A 17** (2002) 627 [hep-ph/0203238] [SPIRES]; *A_4 origin of the neutrino mass matrix*, *Phys. Rev.* **D 70** (2004) 031901 [hep-ph/0404199] [SPIRES]; *Non-abelian discrete symmetries and neutrino masses: two examples*, *New J. Phys.* **6** (2004) 104

[hep-ph/0405152] [SPIRES]; *Non-Abelian discrete family symmetries of leptons and quarks*, hep-ph/0409075 [SPIRES]; *Aspects of the tetrahedral neutrino mass matrix*, *Phys. Rev.* **D 72** (2005) 037301 [hep-ph/0505209] [SPIRES]; *Tetrahedral family symmetry and the neutrino mixing matrix*, *Mod. Phys. Lett.* **A 20** (2005) 2601 [hep-ph/0508099] [SPIRES]; *Tribimaximal neutrino mixing from a supersymmetric model with A_4 family symmetry*, *Phys. Rev.* **D 73** (2006) 057304 [hep-ph/0511133] [SPIRES]; *Suitability of A_4 as a family symmetry in grand unification*, *Mod. Phys. Lett.* **A 21** (2006) 2931 [hep-ph/0607190] [SPIRES]; *Supersymmetric $A_4 \times Z_3$ and A_4 realizations of neutrino tribimaximal mixing without and with corrections*, *Mod. Phys. Lett.* **A 22** (2007) 101 [hep-ph/0610342] [SPIRES];

S.-L. Chen, M. Frigerio and E. Ma, *Hybrid seesaw neutrino masses with A_4 family symmetry*, *Nucl. Phys.* **B 724** (2005) 423 [hep-ph/0504181] [SPIRES];

K.S. Babu and X.-G. He, *Model of geometric neutrino mixing*, hep-ph/0507217 [SPIRES];

A. Zee, *Obtaining the neutrino mixing matrix with the tetrahedral group*, *Phys. Lett.* **B 630** (2005) 58 [hep-ph/0508278] [SPIRES];

G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions*, *Nucl. Phys.* **B 720** (2005) 64 [hep-ph/0504165] [SPIRES]; *Tri-bimaximal neutrino mixing, A_4 and the modular symmetry*, *Nucl. Phys.* **B 741** (2006) 215 [hep-ph/0512103] [SPIRES];

G. Altarelli, F. Feruglio and Y. Lin, *Tri-bimaximal neutrino mixing from orbifolding*, *Nucl. Phys.* **B 775** (2007) 31 [hep-ph/0610165] [SPIRES];

G. Altarelli, F. Feruglio and C. Hagedorn, *A SUSY* SU(5) *grand unified model of tri-bimaximal mixing from A_4*, *JHEP* **03** (2008) 052 [arXiv:0802.0090] [SPIRES];

Y. Lin, *A predictive A_4 model, charged lepton hierarchy and tri-bimaximal sum rule*, *Nucl. Phys.* **B 813** (2009) 91 [arXiv:0804.2867] [SPIRES];

X.-G. He, Y.-Y. Keum and R.R. Volkas, *A_4 flavour symmetry breaking scheme for understanding quark and neutrino mixing angles*, *JHEP* **04** (2006) 039 [hep-ph/0601001] [SPIRES];

B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M.K. Parida, *A_4 symmetry and prediction of $U(e3)$ in a modified Altarelli-Feruglio model*, *Phys. Lett.* **B 638** (2006) 345 [hep-ph/0603059] [SPIRES];

L. Lavoura and H. Kuhbock, *Predictions of an A_4 model with a five-parameter neutrino mass matrix*, *Mod. Phys. Lett.* **A 22** (2007) 181 [hep-ph/0610050] [SPIRES];

S.F. King and M. Malinsky, *A_4 family symmetry and quark-lepton unification*, *Phys. Lett.* **B 645** (2007) 351 [hep-ph/0610250] [SPIRES];

S. Morisi, M. Picariello and E. Torrente-Lujan, *A model for fermion masses and lepton mixing in* SO(10) $\times A_4$, *Phys. Rev.* **D 75** (2007) 075015 [hep-ph/0702034] [SPIRES];

F. Yin, *Neutrino mixing matrix in the* 3-3-1 *model with heavy leptons and A_4 symmetry*, *Phys. Rev.* **D 75** (2007) 073010 [arXiv:0704.3827] [SPIRES];

F. Bazzocchi, S. Kaneko and S. Morisi, *A SUSY A_4 model for fermion masses and mixings*, *JHEP* **03** (2008) 063 [arXiv:0707.3032] [SPIRES];

F. Bazzocchi, S. Morisi and M. Picariello, *Embedding A_4 into left-right flavor symmetry: tribimaximal neutrino mixing and fermion hierarchy*, *Phys. Lett.* **B 659** (2008) 628 [arXiv:0710.2928] [SPIRES];

M. Honda and M. Tanimoto, *Deviation from tri-bimaximal neutrino mixing in A_4 flavor symmetry*, *Prog. Theor. Phys.* **119** (2008) 583 [arXiv:0801.0181] [SPIRES];

B. Brahmachari, S. Choubey and M. Mitra, *The A_4 flavor symmetry and neutrino phenomenology*, *Phys. Rev.* **D 77** (2008) 073008 [*Erratum ibid.* **D 77** (2008) 119901] [arXiv:0801.3554] [SPIRES];

F. Bazzocchi, S. Morisi, M. Picariello and E. Torrente-Lujan, *Embedding A_4 into* SU(3) × U(1) *flavor symmetry: large neutrino mixing and fermion mass hierarchy in SO(10) GUT*, *J. Phys.* **G 36** (2009) 015002 [arXiv:0802.1693] [SPIRES];

P.H. Frampton and S. Matsuzaki, *Renormalizable A_4 model for lepton sector*, arXiv:0806.4592 [SPIRES];

M.-C. Chen and S.F. King, *A_4 see-saw models and form dominance*, *JHEP* **06** (2009) 072 [arXiv:0903.0125] [SPIRES].

[4] C. Hagedorn, M. Lindner and R.N. Mohapatra, *S^4 flavor symmetry and fermion masses: towards a grand unified theory of flavor*, *JHEP* **06** (2006) 042 [hep-ph/0602244] [SPIRES];

C.S. Lam, *Determining horizontal symmetry from neutrino mixing*, *Phys. Rev. Lett.* **101** (2008) 121602 [arXiv:0804.2622] [SPIRES]; *The unique horizontal symmetry of leptons*, *Phys. Rev.* **D 78** (2008) 073015 [arXiv:0809.1185] [SPIRES];

F. Bazzocchi and S. Morisi, *S^4 as a natural flavor symmetry for lepton mixing*, arXiv:0811.0345 [SPIRES];

G. Altarelli, F. Feruglio and L. Merlo, *Revisiting bimaximal neutrino mixing in a model with S^4 discrete symmetry*, *JHEP* **05** (2009) 020 [arXiv:0903.1940] [SPIRES];

H. Ishimori, Y. Shimizu and M. Tanimoto, *S^4 flavor symmetry of quarks and leptons in* SU(5) *GUT*, arXiv:0812.5031 [SPIRES].

[5] A. Blum, C. Hagedorn and M. Lindner, *Fermion masses and mixings from dihedral flavor symmetries with preserved subgroups*, *Phys. Rev.* **D 77** (2008) 076004 [arXiv:0709.3450] [SPIRES].

[6] W. Grimus and L. Lavoura, *A discrete symmetry group for maximal atmospheric neutrino mixing*, *Phys. Lett.* **B 572** (2003) 189 [hep-ph/0305046] [SPIRES];

W. Grimus, A.S. Joshipura, S. Kaneko, L. Lavoura and M. Tanimoto, *Lepton mixing angle $\theta_{13} = 0$ with a horizontal symmetry D_4*, *JHEP* **07** (2004) 078 [hep-ph/0407112] [SPIRES];

H. Ishimori et al., *D_4 flavor symmetry for neutrino masses and mixing*, *Phys. Lett.* **B 662** (2008) 178 [arXiv:0802.2310] [SPIRES]; *Soft supersymmetry breaking terms from $D_4 \times Z_2$ lepton flavor symmetry*, *Phys. Rev.* **D 77** (2008) 115005 [arXiv:0803.0796] [SPIRES];

A. Adulpravitchai, A. Blum and C. Hagedorn, *A supersymmetric D_4 model for μ-τ symmetry*, *JHEP* **03** (2009) 046 [arXiv:0812.3799] [SPIRES].

[7] E. Ma, *Permutation symmetry for neutrino and charged-lepton mass matrices*, *Phys. Rev.* **D 61** (2000) 033012 [hep-ph/9909249] [SPIRES];

J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, *The flavor symmetry*, *Prog. Theor. Phys.* **109** (2003) 795 [*Erratum ibid.* **114** (2005) 287] [hep-ph/0302196] [SPIRES];

S.-L. Chen, M. Frigerio and E. Ma, *Large neutrino mixing and normal mass hierarchy: a discrete understanding*, *Phys. Rev.* **D 70** (2004) 073008 [*Erratum ibid.* **D 70** (2004) 079905] [hep-ph/0404084] [SPIRES];

W. Grimus and L. Lavoura, *$S^3 \times Z_2$ model for neutrino mass matrices*, *JHEP* **08** (2005) 013 [hep-ph/0504153] [SPIRES];

N. Haba and K. Yoshioka, *Discrete flavor symmetry, dynamical mass textures and grand unification*, *Nucl. Phys.* **B 739** (2006) 254 [hep-ph/0511108] [SPIRES];

T. Teshima, *Flavor mass and mixing and S^3 symmetry: an S^3 invariant model reasonable to all*, *Phys. Rev.* **D 73** (2006) 045019 [hep-ph/0509094] [SPIRES];

F. Caravaglios and S. Morisi, *Neutrino masses and mixings with an S^3 family permutation symmetry*, hep-ph/0503234 [SPIRES];

L. Lavoura and E. Ma, *Two predictive supersymmetric $S^3 \times Z_2$ models for the quark mass matrices*, *Mod. Phys. Lett.* **A 20** (2005) 1217 [hep-ph/0502181] [SPIRES];

S. Kaneko, H. Sawanaka, T. Shingai, M. Tanimoto and K. Yoshioka, *Flavor symmetry and vacuum aligned mass textures*, *Prog. Theor. Phys.* **117** (2007) 161 [hep-ph/0609220] [SPIRES];

R.N. Mohapatra, S. Nasri and H.-B. Yu, *S^3 symmetry and tri-bimaximal mixing*, *Phys. Lett.* **B 639** (2006) 318 [hep-ph/0605020] [SPIRES];

Y. Kajiyama, J. Kubo and H. Okada, *D_6 family symmetry and cold dark matter at LHC*, *Phys. Rev.* **D 75** (2007) 033001 [hep-ph/0610072] [SPIRES];

C.D. Carone and R.F. Lebed, *A hexagonal theory of flavor*, *Phys. Rev.* **D 60** (1999) 096002 [hep-ph/9905275] [SPIRES];

N. Haba, A. Watanabe and K. Yoshioka, *Twisted flavors and tri/bi-maximal neutrino mixing*, *Phys. Rev. Lett.* **97** (2006) 041601 [hep-ph/0603116] [SPIRES];

F. Feruglio and Y. Lin, *Fermion mass hierarchies and flavour mixing from a minimal discrete symmetry*, *Nucl. Phys.* **B 800** (2008) 77 [arXiv:0712.1528] [SPIRES].

[8] T. Kobayashi, H.P. Nilles, F. Ploger, S. Raby and M. Ratz, *Stringy origin of non-abelian discrete flavor symmetries*, *Nucl. Phys.* **B 768** (2007) 135 [hep-ph/0611020] [SPIRES].

[9] K.-S. Choi and J.E. Kim, *Quarks and leptons from orbifolded superstring*, Lecture Notes in Physics volume 696, Springer, Berlin Germany (2006).

[10] J.S. Lomont, *Applications of finite groups*, Dover publication, New York U.S.A. (1993).

[11] T. Kobayashi, Y. Omura and K. Yoshioka, *Flavor symmetry breaking and vacuum alignment on orbifolds*, *Phys. Rev.* **D 78** (2008) 115006 [arXiv:0809.3064] [SPIRES].